Chapter 11

Shafting and Associated Parts

A selection of overhead valve camshafts for automotive engines. *Source:* Courtesy of AVL Schrick.

When a man has a vision, he cannot get the power from the vision until he has performed it on the Earth for the people to see.
Black Elk, Oglala Sioux visionary

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This chapter focuses on shafts and related machine elements. Shafts are an essential component of most machines, and are mainly used for power transmission. Shafts are designed to transmit torque and support bending moments and axial loads; they also must be designed so that they do not deflect excessively. This includes dynamic instability as well as static and fatigue loadings. Shafts utilize a number of machine elements to provide functionality. For example, when a torque needs to be transmitted between a shaft and a machine element, that element can be mounted onto the shaft with a key or spline. If a machine element needs to be located at a certain location on a shaft axis, such as against a shoulder, then a retaining ring, set screw, or pin is often used. This chapter also discusses flywheels, which are used to store energy and provide smooth, jerk-free motion. Finally, a wide assortment of coupling types are described. Couplings are used to connect two shafts, and different coupling designs are able to accommodate misalignment and damping of vibration.

Machine elements described in this chapter: Shafts, keys, splines, set screws, retaining rings, flywheels, couplings.
Typical applications: *Shafts:* widespread use to transmit power to a point of operation, automotive crankshafts, cam shafts. *Keys, splines, and set screws:* used to transmit torque to another machine element's hub, such as for gears, pulleys, wire rope drums, mixer agitators, and flywheels. *Retaining rings:* used to fix the axial location of a component, such as rolling element bearing races, pulleys, wheels, bearing sleeves, and gears. *Flywheels:* automotive engines, crushing machinery, milling machinery, and machine tools. *Couplings:* used to connect two shafts, most commonly between the power source or motor shaft and the drive shaft.
Competing machine elements: *Shafts:* gear drives (Chapters 14 and 15), belt and wire rope drives (Chapter 19). *Keys, splines, set screws, retaining rings:* weldments, threaded retainers and adhesive joints (Chapter 16), press and shrink fits (Chapter 10). *Flywheels:* fluid couplings, large gears. *Couplings:* clutches (Chapter 18) and gear drives (Chapters 14 and 15).

Symbols

A	area, m^2
$\tilde{A}$	constant defined in Eq. (11.26)
$\tilde{B}$	constant defined in Eq. (11.27)
C_1	integration constant
C_f	coefficient of fluctuation, Eq. (11.80)
C_t	ring correction factor
c	distance from neutral axis to outer fiber, m
d	diameter, m
d_m	mean spline diameter, m
d_s	set screw diameter, m
E	modulus of elasticity, Pa
g	gravitational acceleration, 9.807 m/s^2
h	height, m
I	area moment of inertia, m^4
I_m	mass moment of inertia, kg-m^2
J	polar area moment of inertia, m^4
K_c	stress concentration factor
K_e	kinetic energy, N-m
K_f	fatigue stress concentration factor
k	spring rate, N/m
k_f	surface finish factor
k_r	reliability factor
k_s	size factor
l	length, m
l_s	spline length, m
M	moment, N-m
M_f	performance index, J/kg
m_a	mass, kg
n	number of teeth
n_s	safety factor
P	normal force, N
P_t	retaining force, N
p	pressure, Pa
p_f	interference pressure, Pa
q_n	notch sensitivity factor
r	radius, m
S_e	modified endurance limit, Pa
S_e'	endurance limit, Pa
S_{se}	shear modified endurance limit, Pa
S_{sy}	shear yield strength, Pa
S_u	ultimate strength, Pa
S_{ut}	ultimate tensile strength, Pa
S_y	yield strength, Pa
T	torque, N-m
T_l	load torque, N-m
T_m	mean torque, N-m
t	time, s
t_h	thickness, m
U	potential energy, N-m
u	velocity, m/s
W	load, N
w	width, m
x, y, z	Cartesian coordinates, m
δ	deflection, m
θ	cylindrical polar coordinate, deg
$\theta_{\omega_{\max}}$	location within a cycle where speed is maximum, deg
$\theta_{\omega_{\min}}$	location within a cycle where speed is minimum, deg
ν	Poisson's ratio
ρ	density, kg/m^3
σ	normal stress, Pa
σ_e	critical stress using distortion-energy theory, Pa
σ_ϕ	normal stress acting on oblique plane, Pa
τ	shear stress, Pa
τ_ϕ	shear stress acting on oblique plane, Pa
ϕ	oblique angle, deg
ω	angular speed, rad/s
ω_ϕ	fluctuation speed, rad/s

Subscripts

a	alternating
c	compression
i	inner
m	mean
o	outer
r	radial
s	shear
θ	circumferential
ω	speed
1,2,3	principal axes

11.1 Introduction

This chapter begins by discussing the design of shafts, making extensive use of the material from Sections 2.8 through 2.12 and Ch. 4 to develop stresses. The failure theories presented in Section 6.7 are used for static failure prediction in Section 11.2; the material in Ch. 7 is used to develop design rules for fatigue of shafts in Section 11.3. Here, combinations of loading are presented, whereas previously each type of loading was considered independently. It is important that this material be understood before proceeding with this chapter. The critical speed of rotating shafts is discussed in Section 11.5. The dynamics and the first critical speed are important, since the rotating shaft becomes dynamically unstable and large vibrations are likely to develop.

Keys, pins, and splines are used to attach devices to a shaft, and are discussed in Section 11.6. These devices use friction or mechanical interference to transmit a torque. Axial position of parts on a shaft can be done with retaining rings, cotter pins, or a number of similar devices. The design of flywheels and couplings are considered in Sections 11.8 and 11.9. Flywheels are valuable energy storage devices that also provide smooth operation. Couplings are used when two shafts need to be connected and are available in a wide variety of forms, the most common of which are presented.

11.2 Design of Shafts for Static Loading

A **shaft** is a rotating or stationary member usually having a circular cross-section much smaller in diameter than in length, and used for power transmission. Machine elements such as gears, pulleys, cams, flywheels, cranks, sprockets, and rolling-element bearings are mounted on shafts, and as such require a well-designed shaft as a prerequisite to their proper function. The loading on the shaft can include combinations of bending (almost always fluctuating); torsion (may or may not be fluctuating); shock; or axial, normal, or transverse forces. All of these types of loading were considered in Chapter 4. Some of the main considerations in designing a shaft are strength, using yield or fatigue (or both) as a criterion; deflection; or the dynamics established by the critical speeds. In general, the shaft diameter will be the variable used to satisfy the design, although in many practical applications the shaft may not have a constant diameter.

Shaft design must consider both static and fatigue failure possibilities. While at first it may appear that considering fatigue only would result in conservative designs, this is not always the case. For example, it is common that a rotating shaft sees predominantly uniform stresses, but on rare occasions encounters a much more significant stress cycle. Cumulative damage as discussed in Section 7.9 can be considered, with Miner's rule as stated in Eq. (7.24) applied to design the shaft. However, the high loading may be a rare event, such as is caused by a machine malfunction or improper operation. Thus, it may occur only extremely rarely, if at all, so that its contribution to fatigue crack growth may be minor.

Even with an overload or malfunction, there is usually some limit to the stresses that are applied to the shaft. For example, the loads can be controlled by using keys or pins (Section 11.6) or slip clutches (Section 18.10). In such circumstances, it is important to make sure that the shaft does not fail statically, especially since the shaft is usually the most difficult component to service or replace. Not surprisingly, shafts are often designed with very large safety factors.

A number of different loading conditions are considered here. Typically, the designer must establish either the minimum shaft diameter to successfully support applied loads or the safety factor for a specific design. This can be done using the approaches in Chapters 2 and 4, but the problem is so common that simplified solutions for each case are presented below.

11.2.1 Bending Moment and Torsion

Bending moments exerted on a shaft produce a maximum stress, from Eq. (4.45), of

$$\sigma_x = \frac{Mc}{I}. \tag{11.1}$$

Similarly, the shear stress due to an applied torque is, from Eq. (4.33),

$$\tau_{xy} = \frac{Tc}{J}, \tag{11.2}$$

where, for a circular cross section,

$$c = \frac{d}{2} \qquad I = \frac{\pi d^4}{64} \qquad \text{and} \qquad J = \frac{\pi d^4}{32}. \tag{11.3}$$

Substituting Eq. (11.3) into Eqs. (11.1) and (11.2) gives

$$\sigma_x = \frac{64Md}{2\pi d^4} = \frac{32M}{\pi d^3}, \tag{11.4}$$

$$\tau_{xy} = \frac{Td/2}{\pi d^4/32} = \frac{16T}{\pi d^3}. \tag{11.5}$$

Note that since $\sigma_y = 0$, these stresses result in a plane stress loading. Therefore, from Eq. (2.16),

$$\sigma_1, \sigma_2 = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}. \tag{11.6}$$

Substituting Eqs. (11.4) and (11.5) into Eq. (11.6) gives

$$\begin{aligned}\sigma_1, \sigma_2 &= \frac{16M}{\pi d^3} \pm \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2} \\ &= \frac{16}{\pi d^3}\left[M \pm \sqrt{M^2 + T^2}\right].\end{aligned} \tag{11.7}$$

From Eq. (2.19), the principal shear stresses are

$$\tau_1, \tau_2 = \pm\sqrt{\tau_{xy}^2 + \left(\frac{\sigma_x}{2}\right)^2}. \tag{11.8}$$

Substituting Eqs. (11.4) and (11.5) into Eq. (11.8) gives

$$\tau_1, \tau_2 = \pm\frac{16}{\pi d^3}\sqrt{M^2 + T^2}. \tag{11.9}$$

Distortion-Energy Theory

As shown in Section 6.7.1 and by Eqs. (6.11) and (6.12), the Distortion-Energy Theory (DET) predicts failure if the von Mises stress satisfies the following condition:

$$\sigma_e = \left(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2\right)^{1/2} = \frac{S_y}{n_s}, \tag{11.10}$$

where S_y is the yield strength of shaft material and n_s is the safety factor. Substituting Eq. (11.7) into Eq. (11.10), the DET predicts failure if

$$\frac{16}{\pi d^3}\left(4M^2 + 3T^2\right)^{1/2} = \frac{S_y}{n_s}. \tag{11.11}$$

Thus, the DET predicts the smallest diameter where failure will occur as

$$d = \left(\frac{32n_s}{\pi S_y}\sqrt{M^2 + \frac{3}{4}T^2}\right)^{1/3}. \tag{11.12}$$

If the shaft diameter is known and the safety factor is desired, Eq. (11.12) becomes

$$n_s = \frac{\pi d^3 S_y}{32\sqrt{M^2 + \frac{3}{4}T^2}}. \tag{11.13}$$

Maximum-Shear-Stress Theory

As shown in Section 6.7.1 and by Eq. (6.8), the Maximum-Shear-Stress Theory (MSST) predicts failure for a plane or biaxial stress state ($\sigma_3 = 0$) if

$$|\sigma_1 - \sigma_2| = \frac{S_y}{n_s}. \tag{11.14}$$

Equation (11.7) gives

$$\frac{32\sqrt{M^2 + T^2}}{\pi d^3} = \frac{S_y}{n_s}. \tag{11.15}$$

Thus, the MSST predicts the smallest diameter where failure will occur as

$$d = \left(\frac{32n_s}{\pi S_y}\sqrt{M^2 + T^2}\right)^{1/3}. \tag{11.16}$$

If the shaft diameter is known and the safety factor is an desired, Eq. (11.16) becomes

$$n_s = \frac{\pi d^3 S_y}{32\sqrt{M^2 + T^2}}. \tag{11.17}$$

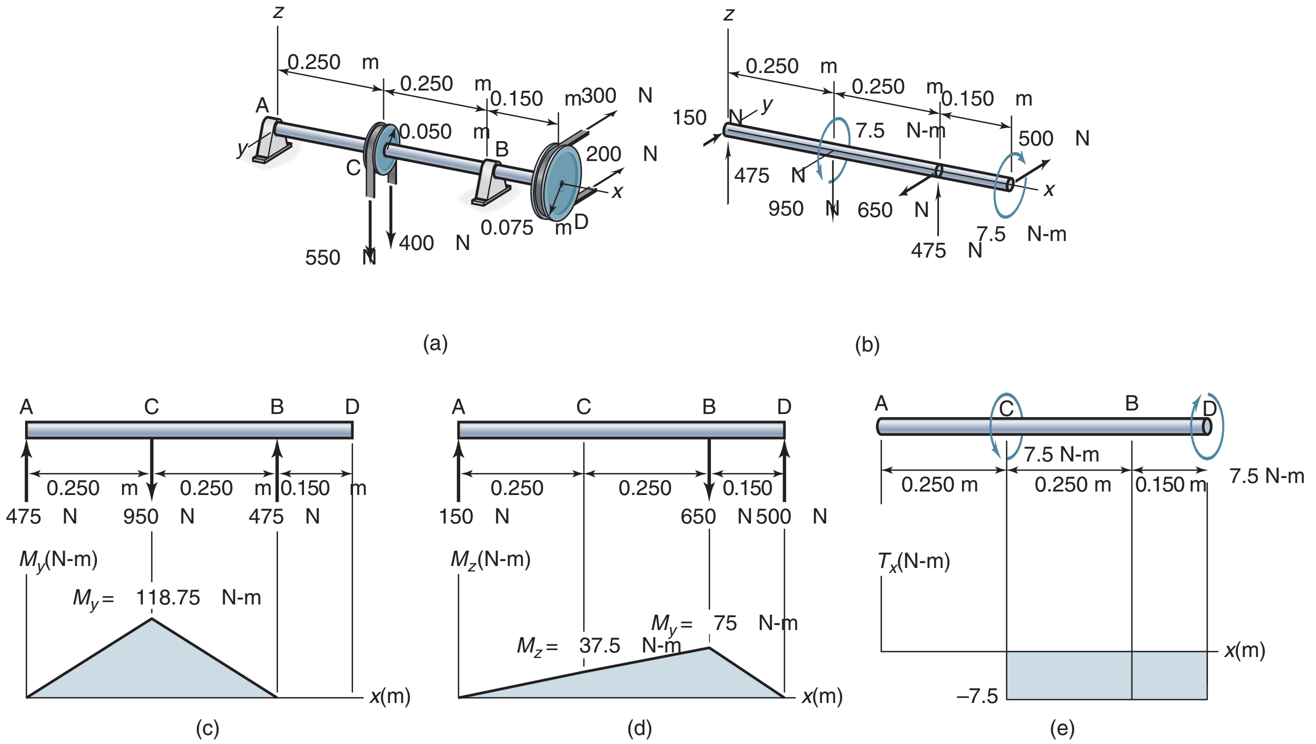

Figure 11.1: Figures used for Example 11.1. (a) Assembly drawing; (b) free-body diagram; (c) moment diagram in x-z plane; (d) moment diagram in x-y plane; (e) torque diagram.

Example 11.1: Static Design of a Shaft

Given: A shaft with mounted belt drives has tensile forces applied as shown in Fig. 11.1a and frictionless journal bearings at locations A and B. The yield strength of the shaft material is 500 MPa.

Find: Determine the smallest safe shaft diameter by using both the DET and the MSST for a safety factor of 2.0. Also, provide a free-body diagram as well as moment and torque diagrams.

Solution: A free-body diagram is shown in Fig. 11.1b; a moment diagram in the x-y plane, in Fig. 11.1c; and a moment diagram in the x-z plane in Fig. 11.1d. These have been constructed using the approach described in Section 2.8. From the moment diagrams, the maximum moment is

$$M_{\max} = \sqrt{(118.75)^2 + (37.5)^2} = 124.5 \text{ N-m}.$$

Figure 11.1e gives the torque diagram. Using the DET, the smallest safe diameter is given by Eq. (11.12) as

$$\begin{aligned} d &= \left(\frac{32 n_s}{\pi S_y} \sqrt{M^2 + \frac{3}{4} T^2} \right)^{1/3} \\ &= \left\{ \frac{32(2)}{\pi(500 \times 10^6)} \left[124.5^2 + \frac{3}{4}(7.5)^2 \right]^{1/2} \right\}^{1/3} \\ &= 17.2 \text{ mm}. \end{aligned}$$

Using the MSST as given in Eq. (11.16) gives

$$\begin{aligned} d &= \left(\frac{32 n_s}{\pi S_y} \sqrt{M^2 + T^2} \right)^{1/3} \\ &= \left\{ \frac{32(2)}{\pi(500 \times 10^6)} \left[124.5^2 + 7.5^2 \right]^{1/2} \right\}^{1/3} \\ &= 17.2 \text{ mm}. \end{aligned}$$

Since the torque is small relative to the moment, little difference exists between the DET and MSST predictions. This is not normally the case, although the results are usually close. Note that it is good design practice to specify a diameter that is rounded up to a convenient integer dimension. In this case, a diameter of 20 mm would be a good option.

11.2.2 Bending, Torsion, and Axial Loading

If, in addition to bending and torsion, an axial load is present, the normal stress is similar to Eq. (11.4) and is given by:

$$\sigma_x = \frac{32M}{\pi d^3} + \frac{4P}{\pi d^2}. \tag{11.18}$$

The shear stress is still expressed by Eq. (11.5); and the principal normal stresses, by Eq. (11.6). Substituting Eqs. (11.18) and (11.5) into Eq. (11.6) gives

$$\begin{aligned} \sigma_1, \sigma_2 &= \frac{16M}{\pi d^3} + \frac{2P}{\pi d^2} \pm \sqrt{\left(\frac{16M}{\pi d^3} + \frac{2P}{\pi d^2} \right)^2 + \left(\frac{16T}{\pi d^3} \right)^2} \\ &= \frac{2}{\pi d^3} \left[8M + Pd \pm \sqrt{(8M + Pd)^2 + (8T)^2} \right]. \end{aligned} \tag{11.19}$$

Substituting Eqs. (11.18) and (11.5) into Eq. (11.8) gives the principal shear stresses as

$$\tau_1, \tau_2 = \pm\frac{2}{\pi d^3}\sqrt{(8M+Pd)^2+(8T)^2}. \tag{11.20}$$

Distortion-Energy Theory

Substituting Eq. (11.19) into Eq. (11.10) shows that the DET predicts failure if

$$\frac{4}{\pi d^3}\sqrt{(8M+Pd)^2+48T^2} = \frac{S_y}{n_s}. \tag{11.21}$$

This equation is more complicated than Eq. (11.16), and an explicit expression for the diameter cannot be obtained. Numerical solutions of Eq. (11.21) are relatively easy to obtain, however.

Maximum-Shear-Stress Theory

Substituting Eq. (11.19) into Eq. (11.14) shows that the MSST predicts failure if

$$\frac{4}{\pi d^3}\sqrt{(8M+Pd)^2+64T^2} = \frac{S_y}{n_s}. \tag{11.22}$$

Again, when an axial loading is included, an explicit expression for the diameter cannot be obtained.

11.3 Fatigue Design of Shafts

In cyclic loading, the stresses vary throughout a cycle and do not remain constant as in static loading. In this section, a general analysis is presented for the fluctuating normal and shear stresses for ductile materials, and appropriate equations are then given for brittle materials. Significant effort is expended in deriving the ultimate expressions for diameter and safety factor, as this makes simplifying assumptions readily apparent. However, the casual reader may wish to proceed to the end of this section where useful design expressions are summarized.

11.3.1 Ductile Materials

Figure 11.2 shows the normal and shear stresses acting on a shaft. In Fig. 11.2a, the stresses act on a rectangular element, and in Fig. 11.2b, they act on an oblique plane at an angle ϕ. The normal stresses are denoted by σ and the shear stresses by τ. Subscript a designates alternating and subscript m designates mean or steady stress. Also, K_f designates the fatigue stress concentration factor due to normal loading, and K_{fs} designates the fatigue concentration factor due to shear loading. On the rectangular element in Fig. 11.2a, the normal stress is $\sigma = \sigma_m \pm K_f\sigma_a$ and the shear stress is $\tau = \tau_m \pm K_{fs}\tau_a$. This uses the approach presented in Section 7.7 that applies the stress concentration factor to the alternating stress and not the mean stress. This approximation has certain implications that will be discussed below.

The largest stress occurs when σ_a and τ_a are in phase, or when the frequency of one is an integer multiple of the frequency of the other. Summing the forces tangent to the diagonal gives

$$\begin{aligned} 0 \quad = \quad & -\tau_\phi A + (\tau_m + K_{fs}\tau_a)A\cos\phi\cos\phi \\ & -(\tau_m + K_{fs}\tau_a)A\sin\phi\sin\phi \\ & +(\sigma_m + K_f\sigma_a)A\cos\phi\sin\phi. \end{aligned}$$

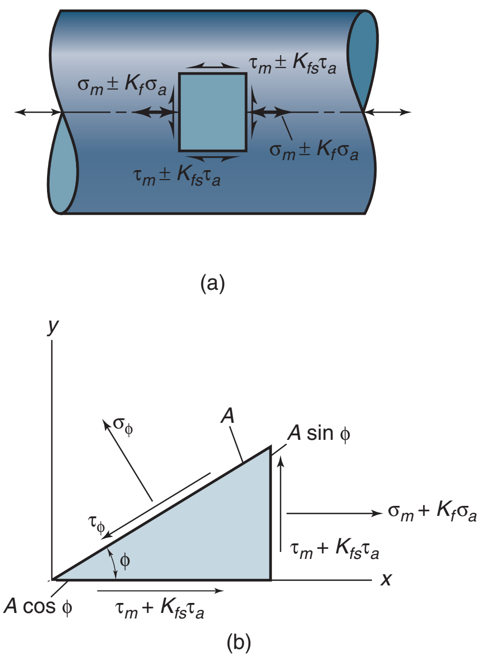

Figure 11.2: Fluctuating normal and shear stresses acting on shaft. (a) Stresses acting on rectangular element; (b) stresses acting on oblique plane at angle ϕ.

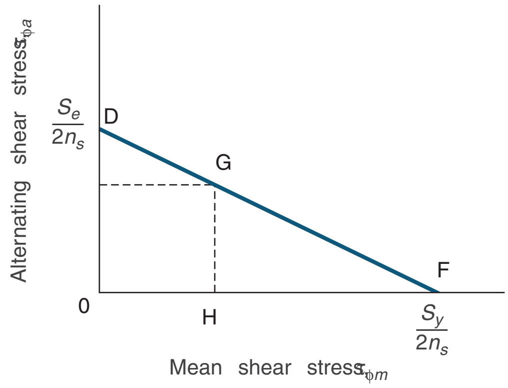

Figure 11.3: Soderberg line for shear stress.

Making use of double angle relations simplifies this expression to

$$\tau_\phi = (\tau_m + K_{fs}\tau_a)\cos 2\phi + \frac{1}{2}(\sigma_m + K_f\sigma_a)\sin 2\phi.$$

Separating the mean and alternating components of stress gives the stress acting on the oblique plane as

$$\begin{aligned} \tau_\phi &= \tau_{\phi m} + \tau_{\phi a} \\ &= \left(\frac{\sigma_m}{2}\sin 2\phi + \tau_m\cos 2\phi\right) \\ &\quad + \left(\frac{K_f\sigma_a}{2}\sin 2\phi + K_{fs}\tau_a\cos 2\phi\right). \end{aligned} \tag{11.23}$$

Recall the Soderberg line in Fig. 7.16 for tensile loading. For shear loading, the end points of the Soderberg line are $S_{se} = S_e/2n_s$ and $S_{sy} = S_y/2n_s$. Figure 11.3 shows the Soderberg line for shear stress. From the proportional triangles GHF and D0F of Fig. 11.3,

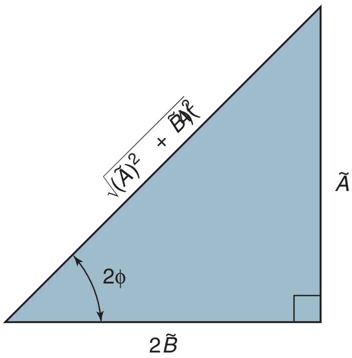

Figure 11.4: Illustration of relationship given in Eq. (11.28).

$$\frac{\mathrm{HF}}{\mathrm{0F}} = \frac{\mathrm{HG}}{\mathrm{0D}}, \qquad \text{or} \qquad \frac{S_y/2n_s - \tau_{\phi m}}{S_y/2n_s} = \frac{\tau_{\phi a}}{S_e/2n_s},$$

so that

$$\frac{1}{n_s} = \frac{\tau_{\phi a}}{S_e/2} + \frac{\tau_{\phi m}}{S_y/2}. \tag{11.24}$$

Substituting the expressions for $\tau_{\phi a}$ and $\tau_{\phi m}$ into Eq. (11.24) gives

$$\begin{aligned} \frac{1}{n_s} &= \frac{\dfrac{K_f\sigma_a}{2}\sin 2\phi + K_{fs}\tau_a\cos 2\phi}{S_e/2} \\ &+ \frac{\left(\dfrac{\sigma_m}{2}\sin 2\phi + \tau_m\cos 2\phi\right)}{S_y/2} \\ &= \tilde{A}\sin 2\phi + 2\tilde{B}\cos 2\phi, \end{aligned} \tag{11.25}$$

where

$$\tilde{A} = \frac{\sigma_m}{S_y} + \frac{K_f\sigma_a}{S_e}, \tag{11.26}$$

$$\tilde{B} = \frac{\tau_m}{S_y} + \frac{K_{fs}\tau_a}{S_e}. \tag{11.27}$$

The stress combination that produces the smallest safety factor is desired, since this corresponds to a maximum-stress situation. The minimum value of n_s corresponds to a maximum value of $1/n_s$. Differentiating $1/n_s$ in Eq. (11.25) and equating the result to zero gives

$$\frac{d}{d\phi}\left(\frac{1}{n_s}\right) = 2\tilde{A}\cos 2\phi - 4\tilde{B}\sin 2\phi = 0.$$

Therefore,

$$\frac{\sin 2\phi}{\cos 2\phi} = \tan 2\phi = \frac{\tilde{A}}{2\tilde{B}}. \tag{11.28}$$

This relationship is illustrated in Fig. 11.4, which shows that

$$\sin 2\phi = \frac{\tilde{A}}{\sqrt{\left(\tilde{A}\right)^2 + 4\left(\tilde{B}\right)^2}}$$

and

$$\cos 2\phi = \frac{2\tilde{B}}{\sqrt{\left(\tilde{A}\right)^2 + 4\left(\tilde{B}\right)^2}}. \tag{11.29}$$

Substituting these into Eq. (11.25) gives

$$\begin{aligned} \frac{1}{n_s} &= \frac{\left(\tilde{A}\right)^2}{\sqrt{\left(\tilde{A}\right)^2 + 4\left(\tilde{B}\right)^2}} + \frac{4\left(\tilde{B}\right)^2}{\sqrt{\left(\tilde{A}\right)^2 + 4\left(\tilde{B}\right)^2}} \\ &= \sqrt{\left(\tilde{A}\right)^2 + 4\left(\tilde{B}\right)^2}. \end{aligned}$$

Substituting Eqs. (11.26) and (11.27) gives

$$\frac{1}{n_s} = \sqrt{\left(\frac{\sigma_m}{S_y} + \frac{K_f\sigma_a}{S_e}\right)^2 + 4\left(\frac{\tau_m}{S_y} + \frac{K_{fs}\tau_a}{S_e}\right)^2},$$

$$\frac{S_y}{n_s} = \sqrt{\left(\sigma_m + \frac{S_y}{S_e}K_f\sigma_a\right)^2 + 4\left(\tau_m + \frac{S_y}{S_e}K_{fs}\tau_a\right)^2}. \tag{11.30}$$

Setting $\sigma_y = 0$, $\sigma_x = \sigma$, and $\tau_{xy} = \tau$ in Eq. (2.19) for biaxial stresses, the maximum shear stress is

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}, \tag{11.31}$$

and the safety factor is

$$n_s = \frac{S_y/2}{\tau_{\max}} = \frac{S_y/2}{\sqrt{(\sigma/2)^2 + \tau^2}} = \frac{S_y}{\sqrt{\sigma^2 + 4\tau^2}},$$

$$\frac{S_y}{n_s} = \sqrt{\sigma^2 + 4\tau^2}. \tag{11.32}$$

Equations (11.32) and (11.30) have the same form, and

$$\sigma = \sigma_m + \frac{S_y}{S_e}K_f\sigma_a \qquad \text{and} \qquad \tau = \tau_m + \frac{S_y}{S_e}K_{fs}\tau_a.$$

Note that the normal and shear stresses each contain a steady and an alternating component, the latter weighted for the effect of fatigue and stress concentration.

By making use of Eqs. (11.4) and (11.5), Eq. (11.30) becomes

$$n_s = \frac{\pi d^3 S_y}{32\sqrt{\left(M_m + \dfrac{S_y}{S_e}K_f M_a\right)^2 + \left(T_m + \dfrac{S_y}{S_e}K_{fs}T_a\right)^2}}. \tag{11.33}$$

If the smallest safe diameter for a specified safety factor is desired, Eq. (11.33) can be rewritten as

$$d = \left[\frac{32n_s}{\pi S_y}\sqrt{\left(M_m + \frac{S_y}{S_e}K_f M_a\right)^2 + \left(T_m + \frac{S_y}{S_e}K_{fs}T_a\right)^2}\right]^{1/3} \tag{11.34}$$

Equations (11.33) and (11.34) represent the general form of a shaft design equation using the Soderberg line and MSST. Note in Eq. (11.34) that S_y, S_y/S_e, K_f, and K_{fs} depend on the shaft diameter d. Thus, a numerical or iterative approach is needed to solve for the required diameter.

Peterson [1974] modified Eq. (11.30) by changing the coefficient of the shear stress term from 4 to 3, such that the DET is satisfied and gives

$$\frac{S_y}{n_s} = \sqrt{\left(\sigma_m + \frac{S_y}{S_e}K_f\sigma_a\right)^2 + 3\left(\tau_m + \frac{S_y}{S_e}K_{fs}\tau_a\right)^2}. \tag{11.35}$$

By making use of Eqs. (11.4) and (11.5), Eq. (11.35) becomes

$$n_s = \frac{\pi d^3 S_y}{32\sqrt{\left(M_m + \frac{S_y}{S_e}K_f M_a\right)^2 + \frac{3}{4}\left(T_m + \frac{S_y}{S_e}K_{fs}T_a\right)^2}}. \tag{11.36}$$

The smallest safe diameter corresponding to a specific safety factor can then be expressed as

$$d^3 = \frac{32 n_s}{\pi S_y}\sqrt{\left(M_m + \frac{S_y}{S_e}K_f M_a\right)^2 + \frac{3}{4}\left(T_m + \frac{S_y}{S_e}K_{fs}T_a\right)^2} \tag{11.37}$$

The distinction between Eqs. (11.33) and (11.34) and Eqs. (11.36) and (11.37) needs to be recognized. Equations (11.33) and (11.34) assume that the MSST is valid; Eqs. (11.36) and (11.37) assume that the DET is valid. All four equations are general equations applicable to ductile materials.

Example 11.2: Fatigue Design of a Shaft

Given: When a rear-wheel-drive car accelerates around a bend at high speeds, the drive shafts are subjected to both bending and torsion. The acceleration torque, T, is reasonably constant at 400 N-m while the bending moment is varying due to cornering and is expressed in newton-meters as

$$M = 250 + 800\sin\omega t.$$

Thus, the mean and alternating moments are $M_m = 250$ N-m and $M_a = 800$ N-m. Assume there is no notch that can produce a stress concentration. The reliability must be 99% and the safety factor is 4.5. The shaft is forged from high-carbon steel, so that it has equivalent mechanical properties as AISI 1080 steel that has been quenched and tempered at 800°C.

Find: The shaft diameter using the MSST.

Solution: From Eq. (7.7) and Table A.2 the bending endurance limit for AISI 1080 steel is

$$S_e' = 0.5S_u = 0.5\,(615) = 307.5 \text{ MPa}.$$

From Fig. 7.11, the surface finish factor for an as-forged surface at $S_{ut} = 615$ MPa is $k_f = 0.42$. To evaluate the size factor, the shaft diameter needs to be chosen, and this value can be modified later if necessary. From Eq. (7.20), and assuming $d = 30$ mm,

$$k_s = 1.248d^{-0.112} = 1.248(30)^{-0.112} = 0.853.$$

From Table 7.4, for 99% probability of survival, the reliability factor is 0.82. Substituting this value into Eq. (7.18) gives the endurance limit as

$$S_e = k_f k_s k_r S_e' = (0.42)(0.853)(0.82)\left(307.5\times 10^6\right),$$

or $S_e = 90.19$ MPa. Substituting into Eq. (11.34) with $T_a = 0$ gives

$$\begin{aligned} d^3 &= \left\{\frac{32n_s}{\pi S_y}\sqrt{\left(M_m + \frac{S_y}{S_e}K_f M_a\right)^2 + T_m^2}\right\}^{1/3} \\ &= \frac{32(4.5)}{\pi\,(380\times 10^6)} \\ &\quad\times\sqrt{\left(250 + \frac{(380\times 10^6)}{(90.19\times 10^6)}(1)(800)\right)^2 + (400)^2}. \end{aligned}$$

which is solved as $d = 0.0760$ m $= 76.0$ mm.
Note that this value is very different from the assumed shaft diameter of 30 mm, so at least one more iteration is required. Use the value of 76.0 mm as the new assumed value for diameter. From Eq. (7.20),

$$k_s = 1.248d^{-0.112} = 1.248(76.0)^{-0.112} = 0.768.$$

Therefore,

$$S_e = (0.42)(0.768)(0.82)(307.5\times 10^6) = 81.33 \text{ MPa}.$$

From Eq. (11.34),

$$\begin{aligned} d^3 &= \frac{32(4.5)}{\pi(380\times 10^6)} \\ &\quad\times\sqrt{\left[250 + \frac{(380\times 10^6)}{(81.33\times 10^6)}(1)(800)\right]^2 + (400)^2}, \end{aligned}$$

or $d = 0.0785$ m. Note that the size factor was calculated based on a diameter of 76.0 mm, and the updated solution is 78.5 mm. Since these are very close, no further iterations are deemed necessary. A diameter of 78.5 mm is an awkward design specification; a reasonable dimension to specify for the shaft would be 80 mm or even larger, depending on such factors as stock availability and cost.

Example 11.3: Fatigue Design of a Shaft Under Combined Loading

Given: The shaft made of AISI 1080 high-carbon steel (quenched and tempered at 800°C) shown in Fig. 11.5 is subjected to completely reversed bending and steady torsion. A standard needle bearing (see Fig. 13.1c) is to be placed on diameter d_2 and this surface will therefore be ground to form a good seat for the bearing. The remainder of the shaft will be machined. The groove between the sections ensures that the large diameter section is not damaged by the grinding operation, and is called a *grinding relief*.

Assume that standard needle bearing bore sizes are in 5-mm increments in the range 15 to 50 mm. Design the shaft so that the relative sizes are approximately (within 1 mm) $d_2 = 0.75d_3$ and $d_1 = 0.65d_3$. At this location, the loading involves completely reversed bending of 70 N-m, and steady torsion of 45 N-m. Design the shaft for infinite life.

Find: Determine the diameter d_2 that results in a safety factor of at least 5.0.

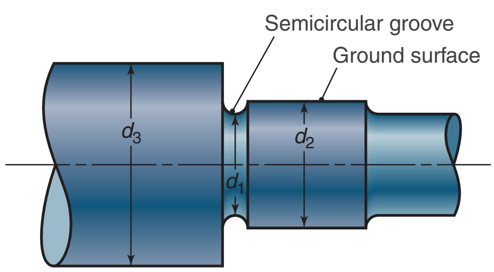

Figure 11.5: Section of shaft in Example 11.3.

Solution: The stress concentration factor can be obtained from the geometry using the ratios given. Not that some variation will occur due to rounding of the dimensions, but this has only a minor effect on the stress concentration. Therefore,

$$\frac{d_2}{d_1} = \frac{0.75d_3}{0.65d_3} = 1.154.$$

Since the grinding relief groove is semicircular,

$$\frac{r}{d} = \frac{(d_2 - d_1)/2}{d_1} = \frac{1}{2}\left(\frac{d_2}{d_1} - 1\right) = 0.0769.$$

Therefore, from Fig. 6.6b, $K_c = 2.05$. From Table A2 for this steel, $S_u = 615$ MPa and $S_y = 380$ MPa. From Eq. (7.7) for bending,

$$S_e' = 0.5S_u = 307 \text{ MPa}.$$

For the shaft, one surface is ground, but the remainder is machined. An inspection of Fig. 11.5 suggests that the ground surface has no stress concentrations and has a larger diameter than the region of grinding relief. Therefore, failure is most likely at the relief, and a surface finish correction factor will be calculated based on a machined surface. Therefore, from Eq. (7.19),

$$k_f = eS_u^f = (4.51)(615)^{-0.265} = 0.822.$$

No yield criterion has been specified, so the MSST will be used. The problem states that the diameter d_2 will be a seat for a bearing, and such bearings are available in 5-mm increments. Therefore, d_2 will be arbitrarily assigned a value of 20 mm, and the safety factor will be calculated and compared to the required value using Eq. (11.33). If the safety factor is not sufficient, then d_2 will be increased until a sufficiently high safety factor results. An alternative approach is to derive an expression for diameter based on Eq. (11.34), and then obtain a numerical solution using a mathematics software package. Either approach is valid and will produce the same results.

If d_2 is 20 mm, then d_3 is $d_2/0.75 = 26.67$ mm, which is rounded up to $d_3 = 27$ mm. Similarly, $d_1 = 0.65d_3 = 17.55$ mm, so that d_1 will be assigned a value of 18 mm. The size factor is obtained from Eq. (7.20) as

$$k_s = 1.248d^{-0.112} = 1.248(27)^{-0.112} = 0.863.$$

The selection of d_3 for use in calculating the size factor should be discussed. No detailed information is given regarding the manufacture of the shaft. It is reasonable to assume that the shaft was machined from extruded bar stock slightly larger than the d_3 dimension, thus justifying the approach in this solution. However, if the shaft were forged, roll forged, or swaged, and then machined to the final dimensions, it would be reasonable to use d_1 to obtain the size factor. Also note that k_f and k_s are coincidentally equal in this case.

No other correction factors apply to this problem, so that the modified endurance limit is obtained from Eq. (7.18):

$$S_e = k_f k_s S_e' = (0.822)(0.863)(307.5) = 218.1 \text{ MPa}.$$

If $d_1 = 18$ and $d_2 = 20$, then the notch radius is $r = 1$ mm for a semicircular groove. Therefore, from Fig. 7.10 for $S_u = 615$ MPa, the notch sensitivity factor is around $q_n = 0.7$. From Eq. (7.17),

$$K_f = 1 + (K_c - 1)q_n = 1 + (2.05 - 1)(0.7) = 1.735.$$

For completely reversed bending, $M_m = 0$ and $M_a = 70$ N-m, and for steady torsion, $T_a = 0$ and $T_m = 45$ N-m. Therefore, from Eq. (11.33),

$$n_s = \frac{\pi(0.020)^3(380 \times 10^6)}{32\sqrt{\left[\left(\frac{380}{218.1}\right)(1.735)(70)\right]^2 + 45^2}} = 1.38.$$

This safety factor is too low, since a minimum safety factor of 5.0 was prescribed. Thus, the diameter d_2 is increased, and the procedure is repeated. Table 11.1 summarizes the results for a number of values of d_2. Therefore, the value of $d_2 = 35$ mm, $d_1 = 31$ mm, and $d_3 = 47$ mm are used to design the shaft.

Table 11.1: Summary of results for Example 11.3.

d_2 (mm)	d_1 (mm)	d_3 (mm)	n_s
20	18	27	1.32
25	22	34	2.50
30	26	40	4.26
35	31	47	6.65

11.3.2 Brittle Materials

Although shafts are usually cold-worked metals that are machined to final desired dimensions, there are applications where castings, which are often brittle materials, are used as shafts. As discussed in Ch. 6, this requires a slightly different analysis approach than for ductile materials.

For brittle materials, the forces in Fig. 11.2b are assumed to be *normal* rather than tangent to the diagonal. Also, the design line for any failure theory relevant to brittle materials (see Section 6.7.2) extends from S_e/n_s to S_u/n_s instead of from $S_e/2n_s$ to $S_y/2n_s$ as was true for the ductile materials. Following procedures similar to those used in obtaining Eq. (11.30) gives

$$\frac{2S_u}{n_s} = K_c\left(\sigma_m + \frac{S_u}{S_e}\sigma_a\right) + \sqrt{K_c^2\left(\sigma_m + \frac{S_u}{S_e}\sigma_a\right)^2 + 4K_{cs}^2\left(\tau_m + \frac{S_u}{S_e}\tau_a\right)^2}. \tag{11.38}$$

where K_c is the theoretical stress concentration factor. By making use of Eqs. (11.4) and (11.5), Eq. (11.38) can be written as

$$n_s = \frac{\pi d^3 S_u/16}{K_c\Psi + \sqrt{K_c^2\Psi^2 + K_{cs}^2\left(T_m + \frac{S_u}{S_e}T_a\right)^2}}, \tag{11.39}$$

where Ψ is given by

$$\Psi = M_m + \frac{S_u}{S_e}M_a. \tag{11.40}$$

If the minimum safe diameter of the shaft is desired for a specific safety factor,

$$d = \left\{ \frac{16 n_s}{\pi S_u} \left[K_c \Psi + \sqrt{K_c^2 \Psi^2 + K_{cs}^2 \left(T_m + \frac{S_u}{S_e} T_a \right)^2} \right] \right\}^{1/3} \tag{11.41}$$

The important difference in the equations developed above for the safety factor and the smallest safe diameter is that Eqs. (11.33) and (11.34) are applicable for ductile materials while assuming the MSST, Eqs. (11.36) and (11.37) are also applicable for ductile materials but while assuming the DET, and Eqs. (11.39) and (11.41) are applicable for brittle materials. Note the major differences between the equations developed for brittle and ductile materials. For brittle materials [Eqs. (11.39) and (11.41)] the stress concentration factor K_c and the ultimate stress S_u are used, whereas for ductile materials [Eqs. (11.33), (11.34), (11.36), and (11.37)] the fatigue stress concentration factor K_f and the yield stress S_y are used.

11.4 Additional Shaft Design Considerations

Sections 11.2 and 11.3 described in detail the design approach for sizing or analyzing a shaft from a stress standpoint. Rotating shafts are very likely to encounter sufficient stress cycles to necessitate design based on an endurance limit (see Section 7.8). Therefore, the same concerns discussed in Ch. 7 hold for shafts. It needs to be recognized that the design approach in Ch. 7 is empirical in nature, but must be verified through experiments. Shafts are often spared from extensive test programs because of their inherently high safety factors, the reasons for which are discussed below.

A common cyclic stress variation that occurs in practical applications is reversed bending and steady torsion. From Section 7.3, note that reversed bending implies that $\sigma_m = 0$, or $M_m = 0$. Also, steady torsion implies that $\tau_a = 0$, or $T_a = 0$. Thus, reduced forms of Eqs. (11.33) and (11.34), (11.36) and (11.37), or (11.39) and (11.41) can be readily determined. For example, for such a loading, Eqs. (11.33) and (11.34) become

$$n_s = \frac{\pi d^3 S_y}{32 \sqrt{\left(\frac{S_y}{S_e} K_f M_a \right)^2 + T_m^2}}, \tag{11.42}$$

$$d = \left[\frac{32 n_s}{\pi S_y} \sqrt{\left(\frac{S_y}{S_e} K_f M_a \right)^2 + T_m^2} \right]^{1/3}. \tag{11.43}$$

Recall that Eqs. (11.42) and (11.43) are for the MSST and Soderberg line; similar expressions can be obtained for other criteria.

As mentioned previously, shafts usually display fairly large safety factors compared to other machine elements. There are a number of reasons for this, including:

1. Shafts are usually in difficult-to-access locations, and have many machine elements mounted onto them. Replacing a shaft requires significant time merely for exposure; removing machine elements, replacing the shaft, and remounting the machine elements (and aligning them) also requires significant time. Recognizing this, designers commonly assign large safety factors to avoid high costs associated with failure and replacement of shafts.
2. Shafts themselves are usually quite expensive, and protection of the shaft is one of the main reasons that keys or pins (Section 11.6) or slip clutches (Section 18.10) are used.
3. Deflection is a major concern, and often shaft size is specified to meet a deflection requirement, leading to low stress levels. Deflection includes lateral deflection of the shaft from bending moments (Ch. 5), as well as torsional deflection (see Section 4.4).
4. Certain machine elements, such as gears or connecting rods, require that the shaft provide load support with minimal deflection. For this reason, it is not unusual to place bearings immediately adjacent to such machine elements. Thus, the spans and bending moments encountered in practice problems and examples are not reflective of well-supported shafts in practice. However, it must be recognized that providing additional bearings is not a straightforward approach, and requires careful alignment and adjustment in order to evenly distribute loads.

Design Procedure 11.1: Shafts

The general procedure for shaft design is as follows:

1. Develop a free-body diagram by replacing the various machine elements mounted on the shaft by their statically equivalent load or torque components. To illustrate this, Fig. 11.6a shows two gears exerting forces on a shaft, and Fig. 11.6b then shows a free-body diagram of the shaft.
2. Draw a bending moment diagram in the x-y and x-z planes as shown in Fig. 11.6c and d. The resultant internal moment at any section along the shaft may be expressed as
$$M_x = \sqrt{M_{xy}^2 + M_{xz}^2}. \tag{11.44}$$
3. Analyze the shaft based on lateral deflection due to bending using the approach in Ch. 5. If deflection is a design constraint, select a diameter that results in acceptable deflection, or else relocate supports to reduce the bending moments encountered.
4. Develop a torque diagram as shown in Fig. 11.6e. Torque developed from one power-transmitting element must balance that from other power-transmitting elements.
5. Analyze the shaft based on deflection due to torsion (Section 4.4). If torsional deflection is a design constraint, select a diameter that results in acceptable deflection.
6. Evaluate the suitability of the shaft from a stress standpoint:

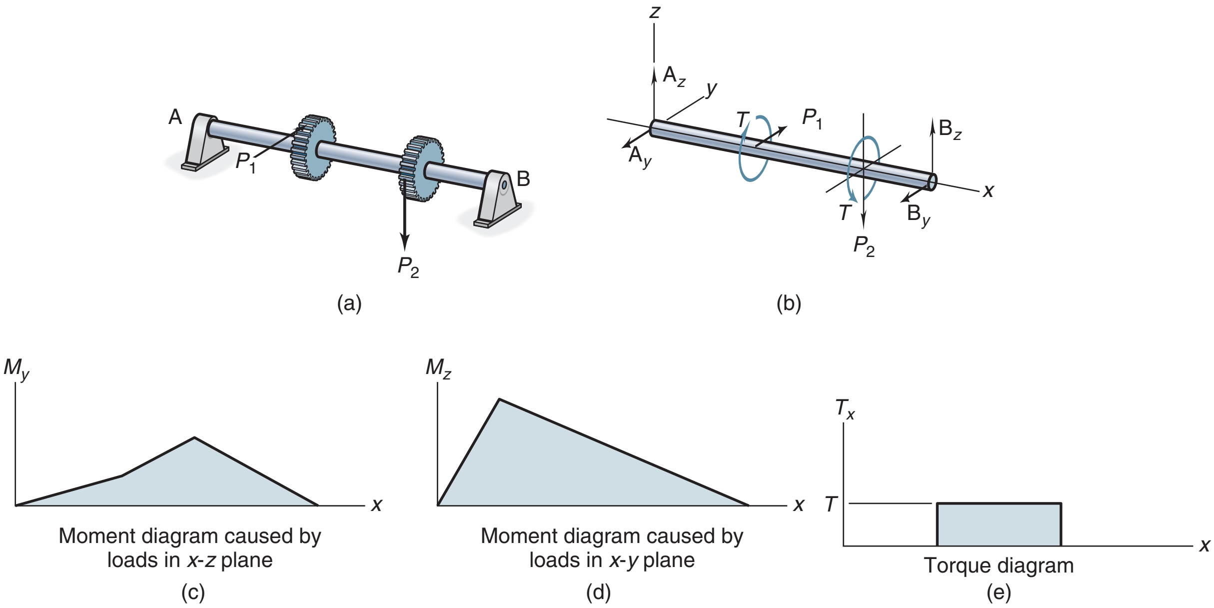

Figure 11.6: Shaft assembly. (a) Shaft with two bearings at A and B and two gears with resulting forces P_1 and P_2; (b) free-body diagram of torque and forces resulting from assembly drawing; (c) moment diagram in x-z plane; (d) moment diagram in x-y planes; (e) torque diagram.

(a) Establish the location of the critical cross-section, or the x location where the torque and moment are the largest.

(b) For ductile materials, use the MSST or the DET covered in Section 6.7.1.

(c) For brittle materials, use the maximum-normal-stress theory (MNST), the internal friction theory (IFT), or the modified Mohr theory (MMT) (see Section 6.7.2).

7. Use keyways, set screws or pins (see Section 11.6), or slip clutches (Section 18.10) where appropriate to protect the shaft.

8. Compare the critical speed of the shaft (Section 11.5) to the operating conditions. Change shaft diameter or supports to avoid critical speeds if necessary.

11.5 Critical Speed of Rotating Shafts

All rotating shafts deflect during operation. The magnitude of the deflection depends on the stiffness of the shaft and its supports, the total mass of the shaft and its attached parts, and the amount of system damping. The **critical speed** of a rotating shaft, sometimes called the **natural frequency**, is the speed at which the rotating shaft becomes dynamically unstable and large deflections associated with vibration are likely to develop. For any shaft there are an infinite number of critical speeds, but only the lowest (first) and occasionally the second are generally of interest to designers. The others are usually so high as to be well out of the operating range of shaft speed. This text considers only the first critical speed of the shaft. Two approximate methods of finding the first critical speed (or lowest natural frequency) of a system are given in this section, one attributed to Rayleigh and the other to Dunkerley.

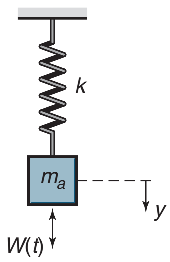

Figure 11.7: Simple single-mass system.

11.5.1 Single-Mass System

The **first critical speed** (or **lowest natural frequency**) can be obtained by observing the rate of interchange between the kinetic (energy of motion) and potential (energy of position) energies of the system during its cyclic motion. A single mass on a shaft can be represented by the simple spring and mass shown in Fig. 11.7. The dashed line indicates the static equilibrium position. The potential energy of the system is

$$U = \int_0^{\delta} (m_a g + k\delta)\, d\delta - m_a g\delta,$$

where

m_a = mass, kg
g = gravitational acceleration, 9.807 m/s^2
k = spring rate, N/m
δ = deflection, m

Integrating gives

$$U = \frac{1}{2}k\delta^2. \tag{11.45}$$

The kinetic energy of the system with the mass moving with a velocity of $\dot{\delta}$ is

$$K_e = \frac{1}{2} m_a \left(\dot{\delta}\right)^2. \qquad (11.46)$$

Observe the following about Eqs. (11.45) and (11.46):

1. As the mass passes through the static equilibrium position, the potential energy is zero and the kinetic energy is at a maximum and equal to the total mechanical energy of the system.
2. When the mass is at the position of maximum displacement and is on the verge of changing direction, its velocity is zero. At this point the potential energy is at a maximum and is equal to the total mechanical energy of the system.

The total mechanical energy is the sum of the potential and kinetic energies and is constant at any time. Therefore,

$$\frac{d}{dt}\left(U + K_e\right) = 0. \qquad (11.47)$$

Substituting Eqs. (11.45) and (11.46) into Eq. (11.47) gives

$$\frac{d}{dt}\left[\frac{1}{2}k\delta^2 + \frac{1}{2}m_a\left(\dot{\delta}\right)^2\right] = k\delta\dot{\delta} + m_a\dot{\delta}\ddot{\delta} = 0.$$

Factoring $\dot{\delta}$ leads to

$$\dot{\delta}\left(m_a\ddot{\delta} + k\delta\right) = 0, \qquad (11.48)$$

$$\ddot{\delta} + \omega^2\delta = 0,$$

where

$$\omega = \sqrt{k/m_a}, \text{ rad/s}. \qquad (11.49)$$

The general solution to this differential equation is

$$\delta = C_1 \sin(\omega t + \phi), \qquad (11.50)$$

where C_1 is an integration constant. The first critical speed (or lowest natural frequency) is ω. Substituting Eq. (11.50) into Eqs. (11.45) and (11.46) gives

$$U = \frac{k}{2}C_1^2 \sin^2\left(\omega t + \phi\right), \qquad (11.51)$$

$$K_e = \frac{m_a}{2}C_1^2\omega^2 \cos^2\left(\omega t + \phi\right). \qquad (11.52)$$

Note from Eq. (11.49) that, for static deflection, if $k = W/\delta$ and $m_a = W/g$, then

$$\omega = \sqrt{\frac{k}{m_a}} = \sqrt{\frac{W/y}{W/g}} = \sqrt{\frac{g}{\delta}}. \qquad (11.53)$$

11.5.2 Multiple-Mass System

From Eq. (11.46), the kinetic energy for n masses is

$$K_e = \frac{1}{2}m_{a1}\left(\dot{\delta}_1\right)^2 + \frac{1}{2}m_{a2}\left(\dot{\delta}_2\right)^2 + \cdots + \frac{1}{2}m_{an}\left(\dot{\delta}_n\right)^2. \qquad (11.54)$$

If the deflection is represented by Eq. (11.50), then $y_{\max} = C_1$. Also, $\dot{y}_{\max} = C_1\omega = y_{\max}\omega$. Therefore, the maximum kinetic energy is

$$K_{e,\max} = \frac{\omega^2}{2}\sum m_{an}\left(\delta_{n,\max}\right)^2. \qquad (11.55)$$

From Eq. (11.45), the potential energy for n masses is

$$U = \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 + \cdots + \frac{1}{2}k_n\delta_n^2, \qquad (11.56)$$

and the maximum potential energy is

$$U_{\max} = \frac{1}{2}k_1\left(\delta_{1,\max}\right)^2 + \frac{1}{2}k_2\left(\delta_{2,\max}\right)^2 + \cdots + \frac{1}{2}k_n\left(\delta_{n,\max}\right)^2. \qquad (11.57)$$

The **Rayleigh method** assumes that $K_{e,\max} = U_{\max}$ or

$$\frac{1}{2}\omega^2 \sum_{i=1,\ldots,n} m_{ai}\left(\delta_{i,\max}\right)^2 = \frac{1}{2}\sum_{i=1,\ldots,n} k_i\left(\delta_{i,\max}\right)^2.$$

Solving for angular velocity,

$$\omega^2 = \frac{\sum_{i=1,\ldots,n} k_i\left(\delta_{i,\max}\right)^2}{\sum_{i=1,\ldots,n} m_{ai}\left(\delta_{i,\max}\right)^2}, \qquad (11.58)$$

but

$$k_i = \frac{W_i}{\delta_{i,\max}} \quad \text{and} \quad m_{ai} = \frac{W_i}{g}. \qquad (11.59)$$

where W_i is the ith weight placed on the shaft and g is gravitation acceleration, 9.807 m/s^2. Substituting Eq. (11.59) into Eq. (11.58) gives

$$\omega_{\text{cr}} = \sqrt{\frac{g\sum_{i=1,\ldots,n} W_i\delta_{i,\max}}{\sum_{i=1,\ldots,n} W_i\delta_{i,\max}^2}}. \qquad (11.60)$$

This is the first critical speed (first natural frequency) of a multiple-mass system when using the Rayleigh method. Equation (11.60) is known as the **Rayleigh equation**. Because the actual displacements are larger than the static displacements used in Eq. (11.60), the energies in both the denominator and the numerator will be underestimated by the Rayleigh formulation. However, the error in the underestimate will be larger in the denominator, since it involves the square of the approximated displacements. Thus, Eq. (11.60) *overestimates* (provides an upper bound on) the first critical speed.

The **Dunkerley equation** is another approximation to the first critical speed of a multiple-mass system; it is given as

$$\frac{1}{\omega_{\text{cr}}^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2}, \qquad (11.61)$$

where

ω_1 = critical speed if only mass 1 exists
ω_2 = critical speed if only mass 2 exists
ω_n = critical speed if only the nth mass exists

Recall from Eq. (11.53) that $\omega_i = \sqrt{g/\delta_i}$.

The Dunkerley equation *underestimates* (provides a lower bound on) the first critical speed. The major difference between the Rayleigh and Dunkerley equations is in the deflections. In the Rayleigh equation, the deflection at a specific mass location takes into account the deflections due to all the masses acting on the system; in the Dunkerley equation, the deflection is due only to the individual mass being evaluated.

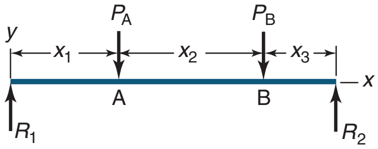

Figure 11.8: Simply supported shaft arrangement for Example 11.4.

Example 11.4: Critical Shaft Speed

Given: Figure 11.8 shows a simply supported shaft arrangement. A solid shaft of 50 mm diameter made of AISI 1020 low-carbon steel is used. The following are given: $x_1 = 0.750$ m, $x_2 = 1.000$ m, $x_3 = 0.500$ m, $P_A = 300$ N, and $P_B = 500$ N.

Find: Determine the first critical speed by using

(*a*) The Rayleigh method

(*b*) The Dunkerley method

Solution: From Table 5.1b, for simply-supported ends, the deflections are given by:

For $0 \le x \le a$

$$\delta_y = \frac{Pbx}{6lEI}(l^2 - x^2 - b^2). \qquad (a)$$

For $a \le x \le l$

$$\delta_y = \frac{Pa(l-x)}{6lEI}(2lx - a^2 - x^2). \qquad (b)$$

From Table 3.1, the modulus of elasticity for carbon steel is 207 GPa. For a solid, round shaft, the area moment of inertia is

$$I = \frac{\pi d^4}{64} = \frac{\pi(0.050)^4}{64} = 3.068 \times 10^{-7} \text{ m}^4.$$

The deflection at location A due to load P_A from Eq. (*a*) and $a = x_1 = x = 0.750$ m and $b = x_2 + x_3 = 1.5$ m is

$$\begin{aligned} \delta_{AA} &= \frac{P_A bx}{6lEI}\left(l^2 - x^2 - b^2\right) \\ &= -\frac{(300)(1.5)(0.75)\left(2.25^2 - 0.75^2 - 1.5^2\right)}{6(2.25)\left(207 \times 10^9\right)\left(3.068 \times 10^{-7}\right)} \\ &= -0.8857 \text{ mm.} \end{aligned}$$

Note that in Fig. 11.8, the y-direction is upward; thus, P_A and δ_{AA} are negative. Also, the first subscript in δ_{AA} designates the location where the deflection occurs, and the second subscript designates the loading that contributes to the deflection. The deflection at location A due to load P_B from Eq. (*a*) and $a = x_1 + x_2$, $x = x_1 = 0.750$ m, and $b = 0.5$ m is

$$\begin{aligned} \delta_{AB} &= \frac{P_B bx}{6lEI}\left(l^2 - x^2 - b^2\right) \\ &= -\frac{(500)(0.5)(0.75)\left(2.25^2 - 0.75^2 - 0.5^2\right)}{6(2.25)\left(207 \times 10^9\right)\left(3.068 \times 10^{-7}\right)} \\ &= -0.9295 \text{ mm.} \end{aligned}$$

The total deflection at location A is

$$\delta_A = \delta_{AA} + \delta_{AB} = -0.8857 - 0.9295 = -1.815 \text{ mm.}$$

The deflection at location B due to load P_B from Eq. (*a*) and $x = a = x_1 + x_2 = 1.75$ m and $b = 0.5$ m is

$$\begin{aligned} \delta_{BB} &= \frac{P_B bx}{6lEI}\left(l^2 - x^2 - b^2\right) \\ &= -\frac{(500)(0.5)(1.75)\left(2.25^2 - 1.75^2 - 0.5^2\right)}{6(2.25)\left(207 \times 10^9\right)\left(3.068 \times 10^{-7}\right)} \\ &= -0.8930 \text{ mm.} \end{aligned}$$

The deflection at location B due to load P_A from Eq. (*b*) and $a = x_1 = 0.75$ m, $b = x_2 + x_3 = 1.5$ m, and $x = x_1 + x_2 = 1.75$ m is

$$\begin{aligned} \delta_{BA} &= \frac{P_A a(l-x)}{6lEI}\left(2lx - a^2 - x^2\right) \\ &= -\frac{(300)(0.75)(2.25 - 2.75)}{6(2.25)\left(207 \times 10^9\right)\left(3.068 \times 10^{-7}\right)} \\ &\quad \times \left[2(2.25)(1.75) - 0.75^2 - 1.75^2\right], \end{aligned}$$

or $\delta_{BA} = -0.5577$ mm. Thus, the total deflection at location B is

$$\delta_B = \delta_{BA} + \delta_{BB} = -0.5577 - 0.8930 = -1.451 \text{ mm.}$$

(*a*) Using the Rayleigh method, Eq. (11.60) gives the first critical speed as

$$\begin{aligned} \omega_{\text{cr}} &= \sqrt{\frac{g(P_A\delta_A + P_B\delta_B)}{P_A\delta_A^2 + P_B\delta_B^2}} \\ &= \sqrt{\frac{(9.81)\left[(300)(0.001815) + (500)(0.001451)\right]}{(300)(0.001815)^2 + (500)(0.001451)^2}}. \end{aligned}$$

or $\omega_{\text{cr}} = 78.13$ rad/s $= 746$ rpm.

(*b*) Using the Dunkerley method, Eq. (11.61) gives

$$\frac{1}{\omega_{\text{cr}}^2} = \frac{1}{\omega_{\text{cr,A}}^2} + \frac{1}{\omega_{\text{cr,B}}^2}.$$

where

$$\omega_{\text{cr,A}} = \sqrt{\frac{g}{\delta_{AA}}} = \sqrt{\frac{9.81}{0.0008857}} = 105.2 \text{ rad/s} = 1005 \text{ rpm},$$

$$\omega_{\text{cr,B}} = \sqrt{\frac{g}{\delta_{BB}}} = \sqrt{\frac{9.81}{0.0008930}} = 104.8 \text{ rad/s} = 1001 \text{ rpm}.$$

Therefore, the critical speed is

$$\frac{1}{\omega_{\text{cr}}^2} = \frac{1}{\omega_{\text{cr,A}}^2} + \frac{1}{\omega_{\text{cr,B}}^2} = \frac{1}{1005^2} + \frac{1}{1001^2},$$

which is solved as $\omega_{\text{cr}} = 709$ rpm.

In summary, the Rayleigh equation gives $\omega_{\text{cr}} = 746$ rpm, which overestimates the first critical speed; the Dunkerley equation gives $\omega_{\text{cr}} = 709$ rpm, which underestimates the first critical speed. Therefore, the actual first critical speed is between 709 and 746 rpm, and the shaft design should avoid this range of operation.